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DEPARTMENT OF COMMERCE * UNITED STATES OF AMERICA *

THE ABC
OF
TRIANGULATION ADJUSTMENT

National Geodetic Survey
Rockville, Md.
March 1977

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H. P. Kaufman, Mathematician
U.S. Coast and Geodetic Survey, 1942

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Survey
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THE A B C OF TRIANGULATION ADJUSTMENT.*

H. P. Kaufman, Mathematician, U. S. C. & G. Survey.

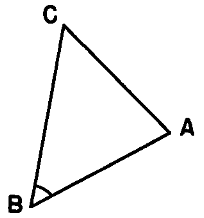
Introduction

In any extensive system of surveys, account must be taken of the curvature of the surface of the earth. In geodetic surveying all calculations are referred to an assumed ideal spheroidal form which is essentially the same as the sea-level surface and closely approximates the actual surface. The position of a point on this ideal surface is determined when it is referred to the system of parallels and meridians, in other words, if its latitude and longitude are known. In the absence of other information the position of a station can be determined only by astronomical observations.

If the positions of two stations are known and the distance between them has been obtained by direct measurement, or is known from previous information, then the position of a third point not in the line joining the other two can be determined if it is observable from both the original stations. It is apparent that the lines connecting the three points form a triangle. The method is known, therefore, as triangulation. It may be extended to include any number of observable stations. By occupying the new stations and observing still other points a system of triangulation is obtained. Such a system may be carried forward to any desired extent and is generally connected with another network. When all determinations are brought into consistent agreement the system is said to be adjusted. The conditions to be satisfied in an adjustment will be considered in connection with the various problems as they are taken up.

Directions

Included in the data obtained by the field parties are lists of directions of occupied stations to observed stations. The direction to a station is the angle taken in a clockwise sense between the line from the occupied station to a station arbitrarily chosen as initial and the line to the given station. There is a list of directions for each occupied station. We shall omit for the time being any discussion of the corrections or reductions which may be necessary and assume that the lists are ready for use. We make use of the directions to determine the angles of the various triangles in a network. In the triangle formed by lines connecting stations A, B, and C, as shown, the angle at B is found by subtracting the direction B to C from B to A. Similarly the angle at C is found by subtracting direction C to A from C to B. It is to be noted that the angle is always taken in a clockwise sense. If the direction to be subtracted is larger than the other, it is necessary to add 360° to the smaller direction before subtracting.



Directions to main scheme stations of first order accuracy are carried to hundredths of a second, intersection stations to tenths or even to whole seconds, if the object sighted on is not sharply defined. Directions to nearby objects can be given only to the nearest 10 seconds.

Triangle Computation, no check Point

Suppose we have the following problem: Stations B and C are occupied. At each, observations are made on station A which is unoccupied. Required to find position of A. It is assumed that we know the latitude and longitude of B and C, the distance B C, the azimuth B to C, and the back azimuth C to B. The azimuth of a line, say B to C, is defined as the angle reckoned clockwise between a line extending due south from B and the line B to C. Similarly the back azimuth is the angle taken clockwise between a line extending due south from C and the line C to B. On a plane surface the forward and back azimuths would differ by 180°, but on the earth north and south lines are meridians and are not parallel except at the Equator. Hence the forward and back azimuth differ by slightly more or less than 180°.

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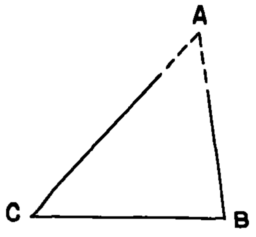
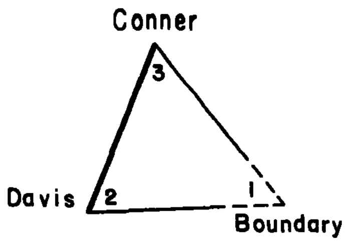
*For additional literature see: Geodesy, Hosmer
U. S. C. & G. S. Spec. Pub. 138
Practical Least Squares, O. M. Leland.

A geodetic line is fixed either by the latitudes and longitudes of both end points or the latitude and longitude of one end point and the length and forward azimuth of the line. That is, if either set of quantities is known the others can be computed.



In the problem suggested let us sketch the triangle, since its form can be deduced from the given data. Lines sighted over in one direction only are shown dotted at the end toward which observation was directed. The angles B and C are obtained from the lists of directions for the points B and C respectively. There is no list for A since the station is unoccupied. Hence the angle A must be concluded. That is A = 180° - (B+C). This is not strictly true since the triangle is not a plane triangle. The sum of the three angles exceeds 180° by a quantity called the spherical excess, the computation of which will be explained subsequently. In the given figure the spherical excess is included in angle A. The spherical excess is computed to as many decimal places as are retained in the directions which fix the angles, in this type of problem usually to tenths of a second. The angles A, B, and C, as determined, are spherical angles. To obtain the sides AC and AB we need to know the plane angles. These are obtained by distributing the spherical excess as nearly equally as possible between the three angles, one third on each angle, or, if it cannot be divided by three, distributed so that the smaller angles will receive their correct share as nearly as possible[*1], and subtracting from each angle its allotted spherical excess. The sum of the three resulting plane angles should be equal exactly to 180°.

The unknown sides BA and CA are next computed using the trigonometric law of sines. This gives us all the data required for the determination of the position of A. The solution of the triangle and the computation of the position of A are carried out on specially prepared forms and will be illustrated by an example taken from actual data.



A rough sketch is drawn to reasonably approximate scale. Following the triangle computation form, the angles are numbered clockwise beginning with the undetermined point. The values of the angles are next entered in the form indicating the concluded angle, which includes all the spherical excess, by parenthesis.

Computation of Triangle

Form 25

NO.	STATION	OBSERVED ANGLE			CORR'N	SPHER'L ANGLE	SPHER'L EXCESS	PLANE ANGLE AND DISTANCE	LOGARITHM
2-3									4.205 406
1	Boundary	(71	07	10.5)		10.5	0.2	10.3	0.024 0190
2	Davis	61	22	16.7		16.7	0.1	16.6	9.943 3673
3	Conner	47	30	33.2		33.2	0.1	33.1	9.867 6947
1-3							0.4		4.172 792
1-2									4.097 120
		180	00	00.4					

The spherical excess is calculated from the formula[*2], a b sin C. x m, involving two sides, the sine of the included angle, and the factor m which depends on the latitude. Since the spherical excess must be determined in advance of the lengths of the sides, we make an approximate preliminary computation of these lengths using the logarithms of the sines of the angles to four places.

[*1] Special Publication No. 138, page 39.

[*2] Special Publication No. 138, pages 32-33.

The steps in the determination of the spherical excess of the triangle, discussed in our example, may be illustrated as follows:

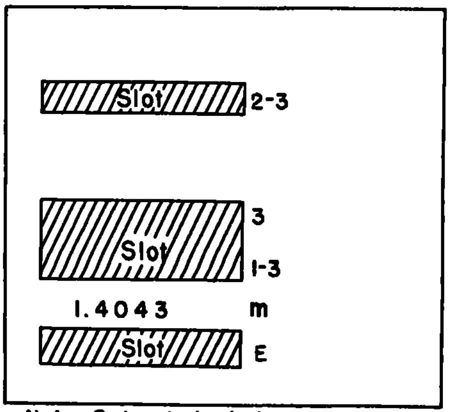
Approximate Computation of Lengths

log (2-3) = 4.2054
colog sin (1) = 0.0240
log sin (2) = 9.9434
log sin (3) = 9.8677
log (1-3) = 4.1728
log (1-2) = 4.0971

Computation of Spherical Excess

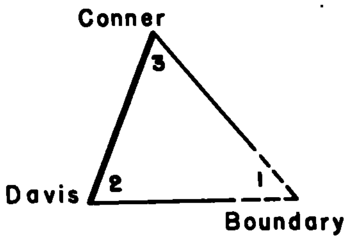
log m = 1.4047
log sin (2) = 9.9434 (= log sin C)
log (1-2) = 4.0971 (= log a)
log (2-3) = 4.2054 (= log b)
log 9.6506
= 0.447
Adopted = 0.4

This will give the spherical excess with sufficient accuracy. Where there is no other means of checking, it is advisable to compute the spherical excess in more than one way since we can use any two sides and the sine of the included angle. For convenience and rapid computation, the spherical excess can be computed from the logarithms which are already in the logarithm column of the triangle computation, by placing over them the shield shown at right. This blocks out all but the necessary logarithms to compute the spherical excess. Log m is usually placed as shown, in pencil, and changed when necessary. The shield uses angle 3 and sides 2-3 and 1-3. The shield can be made of bristol board or other similar paper. The plane angles are next determined. The distribution of spherical excess in the example should be noted. The unknown sides of the triangle are calculated by the trigonometric law of sines. Using the triangle computation form numbering system we have:



Note: Cut out shaded areas.

$$\text{line (1-3)} = \frac{\text{line (2-3) sin (2)}}{\text{sin (1)}}$$



$$\text{line (1-2)} = \frac{\text{line (2-3) sin (3)}}{\text{sin (1)}}$$

or expressed logarithmically

log (1-3) = log (2-3) + colog sin (1) + log sin (2)

log (1-2) = log (2-3) + colog sin (1) + log sin (3)

The form is conveniently arranged for this computation.

The logarithms of lengths are carried to six places and those of functions of angles to seven places, in work of the order of accuracy of the example given.

The position of the point Boundary is next computed using a third order position form. The computation follows the formulas given in Special Publication 8, page 8, making use of only as many terms as the order of accuracy requires. It is to be noted that two independent calculations of the position are made. These will check to within one-thousandth of a second or better, if there is no error in the computation.

A brief explanation of the form will suffice. The latitude and longitude of Boundary may be calculated if we know the latitude and longitude of Davis, the length Davis-Boundary and the azimuth Davis-Boundary. We can also obtain the small correction $\Delta\alpha$ necessary to calculate the back azimuth. A similar statement may be made regarding stations Conner and Boundary. The azimuth Davis-Boundary is obtained by adding the angle at Davis (second angle of the triangle) to the known azimuth Davis-Conner. This gives us α, used as the argument for finding the sine and cosine in the left half of the computation. The constants B, C and D, which are compiled in Special Publication 8, pages 20 to 91, are based upon the latitude φ as an argument.

The right side of the form provides for a similar calculation of the new position using the position of the remaining station, in this example Conner, and the distance and azimuth from it to the new point. The azimuth Conner-Boundary, the α used on the right side, is found by subtracting the angle at Conner from the known azimuth, Conner-Davis. This last statement is made obvious by an inspection of the sketch on page 2. We now have the starting data for both sides of the form. The computation of the increments of latitude, longitude and azimuths, from this point on, follows the same plan on both sides of the form.

In the computation of the increment of latitude $\Delta\varphi$, it is to be noted that substitution in the formula gives $-\Delta\varphi$. If $-\Delta\varphi$ is a positive quantity the latitude of the new station is obtained by subtracting the magnitude of $\Delta\varphi$ from the starting latitude φ. If $-\Delta\varphi$ is negative, the magnitude of $\Delta\varphi$ is added to give us the latitude of the desired position. The sign of the first term of $-\Delta\varphi$ is always the same as that of the sign of $\cos\alpha$. Using the Shortrede Tables this sign is always given, otherwise it must be remembered that $\cos\alpha$ is positive from 0° to 90°, negative from 90° to 270° and positive again from 270° to 360°. The second and third terms are always positive. The rigorous application of the formula for $\Delta\varphi$ requires the computation of a number of terms in addition to the three provided for on the form. Whether they actually need to be brought into the computation depends upon the lengths. The fourth term should be calculated if the logarithm of the length is around 4.0 or more. This term is equal to $-h\, s^2 \sin^2 \alpha\, E$ and is always opposite in sign to the first term. Since the terms are determined to four places of decimals and then adopted to three, the fourth term should always be considered if it amounts to 0".0001. If the values of φ' check within 0".001 the computation is probably correct. The disagreement of 0".001 is arbitrarily removed by adding or subtracting this amount from one of the φ' values. The correction of 0".001 should be applied to that value of φ' for which the corresponding value of $\Delta\varphi$, carried to four places, will be changed least by the correction necessary to alter the value, φ' by 0".001. The determination of $\Delta\lambda'$ is completed after φ' is fixed. Sin α should always be taken out at the same time as $\cos\alpha$. A' is found on the same page of Special Publication 8 as B, C, D and E, but depends on φ' as argument. Since φ' is the same for both sides of the computation, A' and sec φ' are the same in both determinations of $\Delta\lambda$. The formula yields $+\Delta\lambda$. Hence $\Delta\lambda$ is applied with the same algebraic sign as is obtained in the computation, which will be the same as that of $\sin\alpha$. It will be remembered that $\sin\alpha$ is positive from 0° to 180° and negative from 180° to 360°. The two determinations of the longitude λ' should also agree to 0".001 or better. Again an additional correction may be required if the triangle is large. This correction, explained in detail on page 18 of Special Publication 8 and known as the arc sin correction, may be determined using the table on page 17 of Special Publication 8, from the value of s and the preliminary value of $\log \Delta\lambda$. The correction is applied to the logarithm rather than to $\Delta\lambda$ directly.

The value of the longitude as determined on the right and left sides of the form should differ by not more than 1 in the thousandths of seconds, and, as in the case of the latitude, one of the values is adopted. The same considerations are applied in the adoption of λ' as in the adoption of φ'

The importance of following a fairly accurate sketch becomes apparent when it is seen that it enables one to determine by inspection whether $\Delta\varphi$ and $\Delta\lambda$ have been taken in the proper sense.

The computation of the position is completed by the determinations of $-\Delta\alpha$ necessary to give us the back azimuths. For third order work $\log -\Delta\alpha = \log \Delta\lambda + \log \sin \frac{1}{2} (\varphi + \varphi')$. Since we solve for a negative result, $\Delta\alpha$ is always applied with algebraic sign opposite that of $\Delta\lambda$.

After the back azimuths are determined a final check is obtained on the azimuths by adding the angle at the new position to the azimuth 1 to 2. As is apparent from the figure this sum is equal to the azimuth 1 to 3 if correctly determined. We are permitted to adopt one of the back azimuths arbitrarily to correct a discrepancy of not more than 1 in the last place of decimals. It is apparent that the computation is completely self-checking as far as the azimuths are concerned.

In the case of a no-check point the calculation may be entirely correct and

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Form 27
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POSITION COMPUTATION, THIRD-ORDER TRIANGULATION

			°	′	″
α	2 Davis	to 3 Conner	201	54	40.2
2d ∠		&	+ 61	22	16.7
α	2 Davis	to 1 Boundary	263	16	56.9
Δα			+	05	13.7
			180	00	00.0
α′	1 Boundary	to 2 Davis	83	22	10.6 -1
		FIRST ANGLE OF TRIANGLE	71	07	10.5

			°	′	″
α	3 Conner	to 2 Davis	21	57	11.8
3d ∠		&	- 47	30	33.2
α	3 Conner	to 1 Boundary	334	26	38.6
Δα			+	02	42.4
			180	00	00.0
α′	1 Boundary	to 3 Conner	154	29	21.0

	°	′	″			°	′	″
φ	38	00	49.798	2 Davis	λ	75	23	16.340
Δφ	+		47.141		Δλ	-	08	29.242
φ′	38	01	36.939	1 Boundary	λ′	75	14	47.098

	°	′	″			°	′	″
φ	38	08	52.606	3 Conner	λ	75	19	10.404
Δφ	-	07	15.668 +1		Δλ	-	04	23.305
φ′	38	01	36.938	1 1 Boundary	λ′	75	14	47.099-1

	Logarithms	Values in seconds	
s	4.097120		
Cos α	9.0680914 n		
B	8.5110016		
h	1.6762130	1st term	-47.4475″
s²	8.19424		
Sin² α	9.99402		
C	1.29774		
	9.48600	2d term	+0.3062
h²	3.3524	4th	+0.0001
D	2.3792		
	5.7316	3d term	+ 0.0001
		-Δφ	-47.1411

	° ′ ″	
½(φ+φ′)	38 01 13.368	
	Logarithms	**Values in seconds**
s	4.097120	
Sin α	9.9970083 n	
A′	8.5091681	
Sec φ′	0.1036274	
Δλ	2.7069238 +2	-509.2418″
Sin ½(φ+φ′)	9.7895396	
-Δα	2.4964634 +2	-313.6632

	Logarithms	Values in seconds	
s	4.172792		
Cos α	9.9552857		
B	8.5109916		
h	2.6390693	1st term	+435.5814″
s²	8.34558		
Sin² α	9.26974		
C	1.29980		
	8.91512	2d term	+0.0822
h²	5.2781	4th	- 0.0002
D	2.3797		
	7.6578	3d term	+ 0.0045
		-Δφ	+435.6679

	° ′ ″	
½(φ+φ′)	38 05 14.772	
	Logarithms	**Values in seconds**
s	4.172792	
Sin α	9.6348723 n	
A′	8.5091681	
Sec φ′	0.1036274	
Δλ	2.4204598 -3	-263.3054″
Sin ½(φ+φ′)	9.7901890	
-Δα	2.2106488 -3	-162.4235

-h	1.6762 n
$s^2 \sin^2 \alpha$	8.1883
E	6.0635
	5.9280

arc-sin corr.
- 2.8
+ 4.3
+ 1.5

-h	2.6391
$s^2 \sin^2 \alpha$	7.6153
E	6.0663
	6.3207

arc-sin corr.
- 4.0
+ 1.2
- 2.8

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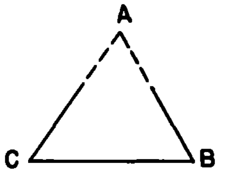
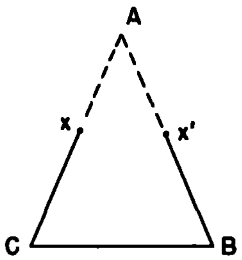
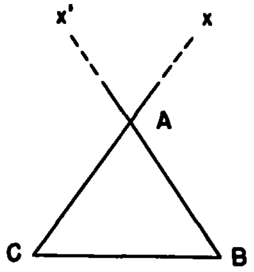
the position yet be spurious. The three adjacent figures illustrate what is meant.





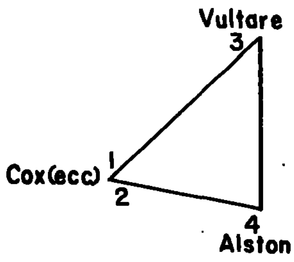


The first indicates a real intersection point at A. Observers at C and B saw the same object. In the other two figures the observers at C and B saw different objects, X and X', identified erroneously as the same. The lines of observation intersect at A and there is nothing in the computation to show that this is a fictitious point. Such an error is liable to occur in a region containing numerous similar objects, such as water tanks or smokestacks.

Check Point with Angle Equation

A point computed from a triangle becomes a check-point, either if the station is occupied and two stations see it or if it is observed from more than two stations. In the first case the three angles of the triangle are determined from observational data and sum to 180° plus the spherical excess with a closing error depending upon the order of accuracy of the work. In the second case the intersection of the lines of observation at a common point may be checked. Furthermore both conditions may be satisfied.

The adjoining figure illustrates the first case. The position of station Cox (ecc.) is to be determined. All stations are occupied. Hence we can determine the error of closure and have one condition to be satisfied in making an adjustment. This condition is called an angle equation. The error of closure is the difference obtained by subtracting the sum of the angles from 180° plus spherical excess. The solution of the angle equation using the theory of least squares gives the most probable distribution of the closure error among the three angles. The numbers placed on the lines leading from the different points are used to indicate the v's or corrections to the directions*. The directions of the two ends of the line Vultare-Alston are fixed by previous adjustment and will receive no correction. Hence no numbers are assigned to this line. The angles are designated as follows; an observer looking into the triangle from any vertex gives the number of the line to the left a negative sign, the one to the right a positive sign. Thus the angle of Cox (Ecc.) is designated - 1+2. At Vultare the angle is +3 since direction Vultare-Alston is unnumbered. Similarly the angle at Alston is designated - 4.



As will be seen more readily from more complicated figures the numbers are taken clockwise about each point beginning with the direction on the extreme left. The scheme of designating angles is consistent with the determination of an angle, that is, the difference between the directions of the lines which form the angle.

Form 25	No.	Station	Observed Angle			Corr'n	Spher'l Angle	Spher'l Excess	Plane Angle and Distance	Logarithm
	2-3	Vultare-Alston								4.139 737
-1+2	1	Cox (ecc)	62	29	26.5	+1.0	27.5	0.1	27.4	0.052 1069
+3	2	Vultare	42	40	31.6	+0.5	32.1	0.1	32.0	9.831 1311
-4	3	Alston	74	50	00.3	+0.5	00.8	0.2	00.6	9.984 6037
	1-3					+2.0		0.4		4.022 975
	1-2									4.176 448
			179	59	58.4					

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*These numerical designations are to be regarded simply as a convenient notation, replacing the familiar x, y, and z, which represent the unknowns in an ordinary algebraic equation. They enter into the equations only as symbols, and their actual values are determined in carrying out the solution.

As soon as the observed angles have been taken out of the list of directions the first three columns of the triangle form No. 25 are filled in. Then the spherical excess is calculated. In the example the spherical excess amounts to 0".4. Since the sum of the three angles amounts to 179° 59' 58.4", the closure is 180° 00' 0.4" - 179° 59' 58.4" or +2.0". We now seek to find the most probable distribution of this closure among the three angles.

The angle equation is:

$$0 = -2.0 - (1) + (2) + (3) - (4)$$

It formally expresses the fact that the sum of the corrections to the angles is equal to the closure. These corrections to the angles are expressed in terms of the corrections to the directions.

The form of the equation is in accordance with the practice of transposing all terms to the right side. This is of no significance in this example, but in cases where several equations are to be solved simultaneously, the constant terms will have the correct algebraic signs, if they appear on the same side of their respective equations as the v's.

The theory of least squares applied to an equation of this type, not solvable uniquely, states that the most probable distribution of errors will be the one for which the sum of the squares of the v's is a minimum. The formal solution is as follows:

$$\begin{aligned}
&(1)\quad 0 = -2.0 - (1) + (2) + (3) - (4)\\
&(2)\quad 0 = -2.0 + 4C\\
&(3)\quad 4C = +2.0\\
&(4)\quad C = +0.5
\end{aligned}$$

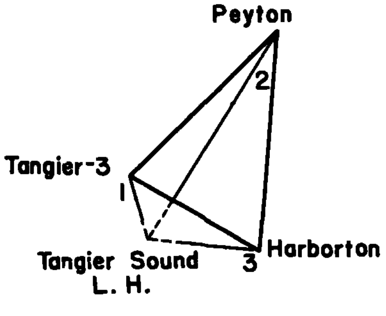
$$v_1 = -0.5,\quad v_2 = +0.5,\quad v_3 = +0.5,\quad v_4 = -0.5$$

The v's are expressed in terms of a constant C. We see, that the equation, $4C = +2$, called a normal equation, is obtained by rewriting the original equation, substituting, for the sum of the v's, the sum of the squares of their coefficients multiplied by C, and equating it to the constant +2.00. The v's are obtained by writing the products of their respective coefficients by C. In an angle equation these coefficients are equal to plus or minus 1. The v's are adopted to the same number of decimal places as the directions in order that the sum of the angle corrections will be exactly equal to the closure of the triangle. In other words the angle equation is satisfied exactly. No adoption is required in the example given since the v's come out exactly to tenths. The steps in the solution are merely indicated here with no attempt at theoretical justification. The angle corrections are determined by combining the v's as shown on the triangle computation form. These angle corrections are applied to the observed angles to give the spherical angles, after which the distributed spherical excess is subtracted yielding the plane angles.

The determination of the unknown sides completes the triangle computation. The corrections to the directions should be applied to the list of directions. The angles of the triangle obtained from the adjusted directions must check the spherical angles in the triangle. The geographic position is finally computed by the method already described, using the spherical angles.

Side Equation

Another relatively simple problem of adjustment arises when an unoccupied point is observed from three or more stations. If the occupied stations have been adjusted already, we have only the condition to be satisfied that the lines drawn from the points of observation to the new station meet in a point. This condition is known as a side equation and will be illustrated by a typical example.



The triangle, Harborton - Tangier 3 - Peyton, is fixed. Hence no numbers are assigned to the lines forming its sides to indicate corrections. The other lines are numbered only at the ends from which observed, since these constitute the only given directions. The designation given a concluded angle, as may be seen from inspection of the

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U. S. COAST AND GEODETIC SURVEY
Form 25
Ed. Jan., 1929

COMPUTATION OF TRIANGLES

State: Virginia

11—0121

NO.	STATION	OBSERVED ANGLE	CORR'N	SPHER'L ANGLE	SPHER'L EXCESS	PLANE ANGLE AND DISTANCE	LOGARITHM
	2-3						4.361603
-1+2	1 Tangier Sd.L.H.	(53 16 20.8)	+1.4	22.2	0.0	22.2	0.0961007
+1	2 Tangier 3	122 43 59.7	-0.2	59.5	0.1	59.4	9.9248982
-2	3 Peyton	3 59 39.6	-1.2	38.4	0.0	38.4	8.8429337
	1-3						4.382602
	1-2						3.300637
					0.1		
	2-3						4.306754
-1+3	1 Tangier Sd.L.H.	(156 40 27.1)	-1.6	25.5		25.5	0.4023418
+1	2 Tangier 3	21 05 19.5	-0.2	19.3		19.3	9.5560765
-3	3 Harborton	2 14 13.4	+1.8	15.2		15.2	8.5915408
	1-3						4.265172 +1
	1-2						3.300637
					0.0		
	2-3						4.526055
-2+3	1 Tangier Sd.L.H.	(103 24 06.3)	-3.0	03.3	0.3	03.0	0.0119887
+2	2 Peyton	32 14 29.7	+1.2	30.9	0.4	30.5	9.7271292
-3	3 Harborton	44 21 25.1	+1.8	26.9	0.4	26.5	9.8445589
	1-3						4.265173
	1-2						4.382602
					1.1		
	2-3						
	1						
	2						
	3						
	1-3						
	1-2						

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triangle computation form, is the sum of the designations given the other two angles of the same triangle but with the signs changed.

The data enable us to fill in the first three columns of the triangle form, except for the concluded angles which must include the spherical excess for the whole triangle in each case. The spherical excess for each triangle is computed using the method already described. We can check the accuracy of the result, since the spherical excess of the fixed triangle is known already. The spherical excess is a function of area and therefore the sum of the spherical excesses for each pair of triangles which form the entire quadrilateral must be equal to that of the other pair. Furthermore, the concluded angle Harborton, Tangier Sd. L.H., Tangier 3, must be equal to the sum of the two smaller concluded angles as indicated by the figure. Realization of this condition may require an adoption of the value of the spherical excess to tenths of a second, not exactly in accord with the computed figure carried to hundredths. In any case both the conditions for the sum of spherical excesses and for the sum of the concluded angles must be satisfied, before going on with the problem.

It should be noted also when the computation for spherical excess is made, whether the logarithms of the sides common to the various triangles come out nearly equal in the computation of the triangles. If this check fails, when the angles are known to be taken correctly from the directions, the point whose position is sought is not a genuine intersection point and it is useless to proceed further with the adjustment. If there are directions available from more than three stations it may still be possible to find three lines which give a real intersection, in which case it is necessary to eliminate the spurious direction and work from another set of 3 directions. If, after using all available directions, it is still impossible to find a definite intersection, the point must be rejected altogether.

Assuming that all checks are satisfactory we may proceed with the set up and solution of the condition equation, which assures that the lines intersecting at the point, Tangier Sd. L. H., actually meet at a point. The following equation may be seen to hold between the lines intersecting at the point, Tangier Sd. L.H.

$$\frac{\text{Tangier Sd. L. H. - Tangier 3}}{\text{Tangier Sd. L. H. - Peyton}} \text{ X } \frac{\text{Tangier Sd. L.H. - Peyton}}{\text{Tangier Sd. L. H. - Harborton}} \text{ X } \frac{\text{Tangier Sd. L, H. - Harborton}}{\text{Tangier Sd. L. H. - Tangier 3}} = 1$$

According to the trigonometric law of sines a side of a triangle is proportional to the sine of the opposite angle. Hence we may substitute sines of angles for the corresponding opposite sides and obtain:

$$\frac{\sin(-2)}{\sin(+1)} \times \frac{\sin(-3)}{\sin(+2)} \times \frac{\sin(+1)}{\sin(-3)} = 1$$

When this equation is expressed logarithmically, the difference between the sums of logarithms in the numerator and denominator, respectively, should be equal to zero. This will be true except for a small residual term, called the constant term, which is to be used in determining the v's. The equation is written on the form especially arranged for the solution.

	1.	2.	3.	4.	5.	6.	7.	8.
			+ Side				- Side	
	-2	3 59 39.6	8.842 9698	+30.17	+1	122 43 59.7	9.924 8978	-1.35
	-3	44 21 25.1	9.844 5558	+ 2.15	+2	32 14 29.7	9.727 1265	+3.34
	+1	21 05 19.5	9.556 0776	+ 5.46	-3	2 14 13.4	8.591 4438	+53.90
			8.243 6032				8.243 4681	

$$\begin{array}{r} +\ 8.243\ 6032 \\ -\ 8.243\ 4681 \\ \hline +\ 1351 \end{array}$$

$0 = +\ 135.1 + 6.81(1) - 33.51(2) + 51.75(3)$

$0 = +\ 135.1 + 3847.3587C$

$3847.3587C = -\ 135.1$

$C = -\ 0.035115$

v1 = - 0.2

v2 = + 1.2

v3 = - 1.8

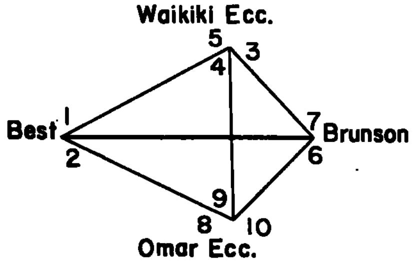
Columns 1 and 5 contain the designations of the angles in the numerator and denominator respectively, columns 2 and 6, the observed angles, column 3 and 7, the logarithms of the sines, and columns 4 and 8 the tabular difference in the logarithm for an increment of 1 second, expressed in units of the sixth place.* The sums of the logarithms in columns 3 and 7 are obtained and the difference of the sums, also expressed in units of the sixth place, noted. This difference is the constant term of the equation and is plus or minus depending on whether the left or right column is the greater in magnitude.

In judging whether the constant term is small enough to give a satisfactory adjustment, account must be taken both of the allowable precision in the angle determinations, and the magnitude of the angles themselves. An error of 1 second in the angle will affect the logarithm and hence the constant term by the amount of the tabular difference for that angle. For small angles the tabular difference becomes very large. In the example shown the constant term is not too large because the equation contains two small angles. The tabular differences in column (4) are written as coefficients of the numbers designating the v's in column (1), the proper algebraic sign being given to the indicated products. Similarly multiplication of the numbers in column (5) by the tabular differences in column (8) is indicated but with algebraic sign changed. The terms so obtained are assembled with the constant term, and the sum set equal to zero. This equation is similar to the angle equation already considered except that the coefficients of the v's in general differ from unity. The solution is handled in exactly the same way. The sum of the squares of the coefficients of the v's multiplied by C is equated to the constant term. After C is solved, it is multiplied by the respective coefficients in the assembled equation to give the v's. The v's are rounded off to the same number of decimals to which the angles are expressed. The values should be adopted to give the most consistent triangle computation and the last figure may not be exactly the same as that arrived at in dropping the extra decimals. This is called adoption of the v's. Because one angle is concluded the sum of the angle corrections is zero for each triangle.

The corrections may now be applied to give the spherical angles and the distributed spherical excess subtracted to give the plane angles. The plane angles should sum to 180° exactly for each triangle. The unknown sides of the triangles are finally computed. Those common to different triangles should be equal or differ in the logarithm by not more than the tabular difference corresponding to 1 in the last decimal place retained in expressing the angles. In intersection work the lengths are rounded off to six places. If common lengths obtained from different triangles are not exactly consistent that one is adopted which is determined from the triangle in which the length is computed from the largest length angles. By length angles are meant the angles of the triangle opposite the sides through which the length is calculated. After the adjustment is completed, the position computation may be made using any one of the triangles containing the unknown point.

Adjustment of a Quadrilateral

The adjustments so far considered have been special cases involving the solution of either a single angle equation or a single side equation. The adjustment of networks involving four or more stations will require, in general, the simultaneous solution of both types of equations. A typical case is that of a quadrilateral with all stations occupied. We choose an example in which two stations have been fixed by previous adjustments. In this case the maximum number of equations are obtained. In the figure the scheme already explained for designating angles has been followed. No numbers are placed on the fixed line. The angles are computed from the list of directions and the first three columns of the triangle form filled out. The spherical excesses are computed and must be the same for the entire quadrilateral using either pair of triangles which form the entire area. The closures are next determined in the manner already discussed. As in the case of the spherical excesses the sum of the closures for the entire area is the same when taken over either of the pairs of triangles which exactly cover the figure. These two checks must be satisfied before proceeding with the adjustment.



A graphical method of determining the number of angle and side equations will be discussed. We shall refer to this as the method of building up the figure

* Being plus for angles less than 90° and minus for angles more than 90°.

DEPARTMENT OF COMMERCE
U. S. COAST AND GEODETIC SURVEY
Form 25
Ed. Jan., 1929

COMPUTATION OF TRIANGLES

State: Virginia

11—0121

	NO.	STATION	OBSERVED ANGLE	CORR'N	SPHER'L ANGLE	SPHER'L EXCESS	PLANE ANGLE AND DISTANCE	LOGARITHM
		2-3						4.2334927
-	8+10	1 Omar Ecc.	113 41 53.60	-0.92	52.68	0.07	52.61	0.03825770
	+2	2 Best	20 39 38.75	+0.30	39.05	0.07	38.98	9.54757192
	-6	3 Brunson	45 38 29.46	-0.98	28.48	0.07	28.41	9.85429141
		1-3		-1.60		0.21		3.8193223
		1-2	01.81					4.1260418
		2-3						3.8193223
-	3+4	1 Waikiki Ecc.	39 47 17.48	-0.79	16.69	0.04	16.65	0.19385517
-	6+7	2 Brunson	96 09 51.26	-0.47	50.79	0.04	50.75	9.99748182
-	9+10	3 Omar Ecc.	44 02 53.16	-0.52	52.64	0.04	52.60	9.84214728
		1-3		-1.78		0.12		4.0106593
		1-2	01.90					3.8553248
		2-3						4.2334927
	-3+5	1 Waikiki Ecc	105 42 42.29	+0.82	43.11	0.08	43.03	0.01653823
	+7	2 Brunson	50 31 21.80	+0.51	22.31	0.08	22.23	9.88754872
	-1	3 Best	23 45 54.69	+0.13	54.82	0.08	54.74	9.60529383
		1-3		+1.46				4.1375796
		1-2	58.78			0.24		3.8553248
		2-3						4.1260418
	-4+5	1 Waikiki Ecc.	65 55 24.81	+1.61	26.42	0.11	26.31	0.03952686
	-8+9	2 Omar Ecc.	69 39 00.44	-0.40	00.04	0.11	59.93	9.97201098
	-1+2	3 Best	44 25 33.44	+0.43	33.87	0.11	33.76	9.84509062
		1-3		+1.64		0.33		4.1375796
		1-2	58.69					4.0106593

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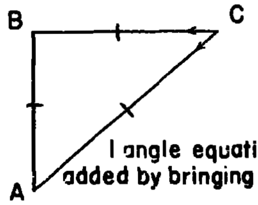
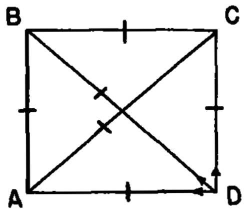
point by point. It is especially advantageous in that it may be applied to a network of any degree of complexity. Suppose two stations A and B are fixed by previous adjustment. Connect these by a heavy line. Next connect the third point C to points A and B. This gives one triangle and one angle equation. Mark each side of the triangle C A B by a short dash. All three sides of the figure are marked indicating that all conditions have been used, which in this case gives one angle equation. Proceeding to the fourth point D, connect it by lines to the three fixed points completing the quadrilateral. Then an angle equation is written in the triangle D B C, the lines B D and C D are marked. This leaves only one line A D unmarked. The last angle equation may be written in either the triangle D A B or D A C to satisfy the last angle condition. Thus two angle equations are added by bringing in the point D, making three in all for the quadrilateral. Stated otherwise the rule is that each new angle equation written in a triangle must contain at least one full line not previously used. It will be noted that the inclusion of a new point permits the introduction of one less angle equation than the number of full lines drawn to the new point from points already used. In practice it is not necessary to actually draw these lines. The figure already drawn should be built up mentally point by point, introducing new lines, but these lines need not be drawn.



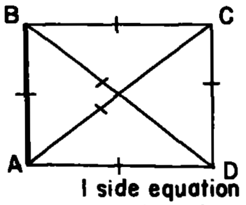
1 angle equation added by bringing in point C



2 angle equations added by bringing in point D

Since a quadrilateral with both diagonals drawn consists of four triangles, it is possible to write four angle equations, of which, according to the rule just developed, only three may be retained. The fact that the four equations are not independent may be seen from the example given. Referring ahead to the three angle equations selected, it is seen that if the first equation is subtracted from the sum of the second and third exactly the same equation is obtained as that derived from the remaining triangle. If all four angle equations were retained the solution of the normal equations would fail due to the appearance of an identical equation. Whereas the rule limits the number of equations it leaves us some freedom of selection. In a simple quadrilateral any three of the four angle equations may be retained but is it advisable to omit the one derived from the triangle containing the smallest angle.

The graphical method will now be applied to the determination of the number of side equations. Completion of the triangle C A B will not yield a side equation, since the intersection of three lines is required. Hence we proceed to D and complete the entire figure. Let us choose point D, the point of intersection of A D, B D, and C D as the pole. The angles used in writing the equation are taken from triangles D A C, D A B, and D B C. If the sides of these triangles are marked, there remain no unmarked lines in the figure. Hence only one side equation may be written. Choice of any other point as pole likewise would have exhausted all available conditions. If a figure is built up point by point, it will be found that, for each new point included, there are introduced two less side equations than the number of lines drawn from the new point to points previously used. Each new equation written must bring in one or more lines not previously used.



1 side equation brought in by point D

It is advantageous, as the figure is built up to keep a record of stations added, lines drawn, and resulting equations, in tabular form. In the case of the example under discussion this table would appear as follows:

1 Name of Station	2 No. lines drawn	3 No. angle equations	4 No. side equations
Brunson	0	0	0
Best	1	0	0
Waikiki ecc.	2	1	0
Omar ecc.	3	2	1
Totals 4	6	3	1

Inspection of this record will indicate either the number of equations introduced with each new point brought into the figure or, by adding the columns, the total number of equations up to the given point. This scheme also gives the most ready access to the data for substitution in the general formulas for determining the number of equations:

Number of angle equations = n'- S'+ 1
Number of side equations = n - 2S + 3.

in which n is the total number of lines, S'is the total number of stations, n' is the number of lines sighted over in both directions and S' is the number of occupied stations. In order to use these formulas we must include the adjusted points and connecting lines which constitute the starting data in building the figure. The quantity, S, is found by counting the names in the first column, whereas n is the sum of the numbers in the second. The number of angle or side equations is given by the sums of the numbers in the third and fourth columns, respectively, and should check the results given by the respective formulas. It is here assumed that all stations are occupied and all lines sighted both ways so that n = n' and S = S'. Otherwise it would be necessary to indicate unoccupied stations and introduce an additional column for lines sighted both ways. The usefulness of this tabular scheme, is enhanced in the case of extended arcs of triangulation or complex net works.

There is usually some freedom of choice in selecting the side equation or equations to satisfy the rule just given. In a simple quadrilateral with two diagonals drawn the equation may be written in one of four possible ways. It is preferable to choose the pole so that the smallest angles will be used in the side equation.* The angle equations and side equation which were chosen are as follows:

Angle Equations.

1. 0 = -1.46 -(1) -(3) +(5)+(7)
2. 0 = -1.64 -(1) +(2) -(4)+(5) -(8)+(9)
3. 0 = +1.78 -(3) +(4) -(6)+(7) -(9)+(10)

Side Equation

		+				-	
-4+5	65 55 24.81	9.96047172	+0.94	-1+2	44 25 33.44	9.84508994	+2.15
-6+7	96 09 51.26	9.99748170	-0.23	-3+4	39 47 17.48	9.80614693	+2.53
+2	20 39 38.75	9.54757063	+5.58	-6	45 38 29.46	9.85429357	+2.06
		9.50552405				9.50553044	

4. 0= -6.39 + 2.15(1) +3.43 (2) + 2.53 (3) -3.47(4) + 0.94(5)
+2.29(6) - 0.23 (7)

Correlate Equations

There are thus obtained four equations containing ten unknowns, which are the v's or corrections to directions. The assembly of correlate equations from these four equations, the formation and solution of the normal equations and the final determination of the v's are given completely in the following pages. The steps taken will be pointed out in order and in detail but with no attempt at a theoretical explanation. In the correlate equations are assembled the coefficients of the v's from each of the original equations, for each of which there is one vertical column. There is a horizontal row corresponding to each v. The correlates are filled in vertically one column at a time. It will be noticed that the columns contain the coefficients of the v's each placed opposite its respective v as indicated in the left hand margin.

*Special Publication No. 138, page 35.

Correlate Equations

-----Equations------

	1	2	3	4	Σ
v_1	-1	-1		+2.15	+0.15
v_2		+1		+3.43	+4.43
v_3	-1		-1	+2.53	+0.53
v_4		-1	+1	-3.47	-3.47
v_5	+1	+1		+0.94	+2.94
v_6			-1	+2.29	+1.29
v_7	+1		+1	-0.23	+1.77
v_8		-1			-1.00
v_9		+1	-1		0.00
v_{10}			+1		+1.00

Normal Equations

	1	2	3	4	η	Σ
(1)	+4	+2	+2	-3.97	-1.46	+2.57
(2)		+6	-2	+5.69	-1.64	+10.05
(3)			+6	-8.52	+1.78	-0.74
(4)				+41.0098	-6.39	+27.8198

Forward Solution

(row)	1	2	3	4	η	Σ
(1)	+4.0000	+2.0000	+2.0000	-3.9700	-1.4600	+2.5700
(2)	C1	-0.50000	-0.50000	+0.99250	+0.36500	-0.64250
(3)		+6.0000	-2.0000	+5.6900	-1.6400	+10.0500
(4)	(1)	-1.0000	-1.0000	+1.9850	+0.7300	-1.2850
(5)		+5.0000	-3.0000	+7.6750	-0.9100	+8.7650
(6)		C2	+0.60000	-1.53500	+0.18200	-1.75300
(7)			+6.0000	-8.5200	+1.7800	-0.7400
(8)		(1)	-1.0000	+1.9850	+0.7300	-1.2850
(9)		(2)	-1.8000	+4.6050	-0.5460	+5.2590
(10)			+3.2000	-1.9300	+1.9640	+3.2340
(11)			C3	+0.60312	-0.61375	-1.01062
(12)				+41.0098	-6.39	+27.8198
(13)			(1)	- 3.9402	-1.4490	+ 2.5507
(14)			(2)	-11.7811	+1.3968	-13.4543
(15)			(3)	- 1.1640	+1.1845	+ 1.9505
(16)				+24.1245	-5.2577	+18.8667
(17)				C4	+0.21794	- 0.78206

Back Solution

(row)	4	3	2	1
(1)	+0.21794	-0.61375	+0.18200	+0.36500
(2)		+0.13144	-0.33454	+0.21631
(3)		-0.48231	-0.28939	+0.24116
(4)			-0.44193	+0.22096
(5)				+1.04343

Determination of v's

v_1	v_2	v_3	v_4	v_5
- 1.04343	-0.44193	-1.04343	+0.44193	+1.04343
+ 0.44193	+0.74753	+0.48231	-0.48231	-0.44193
+ 0.46857		+0.55139	-0.75625	+0.20486
- 0.13293	+0.30560	-0.00973	-0.79663	+0.80636
- 0.13	+0.30	-0.01	-0.80	+0.81

v_6	v_7	v_8	v_9	v_{10}
+ 0.48231	+1.04343	+0.44193	-0.44193	-0.48231
+ 0.49908	-0.48231		+0.48231	
	-0.05013			
+ 0.98139	+0.51099	+0.44193	+0.04038	-0.48231
+ 0.98	+0.51	+0.44	+0.04	-0.48

The first three columns of the correlate equations correspond to the three angle equations, the fourth to the side equation while the fifth or Σ column contains the sums of the horizontal rows.

Normal Equations

We now have the data properly arranged for the formation of the normal equations. In this case there will be four equations corresponding to the numbered columns of the correlates. The first term of the first equation is the sum of the squares of the quantities in the first column of correlates. The second term of the first equation is the sum of the products of quantities in the same horizontal rows of the first and second columns of correlates. Thus in the first row we have (-1) x (-1) = +1 in the fifth, (+1) x (+1) = +1, giving a sum, +2. The third term is the sum of the products of quantities in the same horizontal rows of the first and third column, and the fourth term is obtained similarly from the first and fourth column.

Blanks in the correlate columns are treated as zeros. The η term is the constant term of the corresponding original equation. We note that the first η term, -1.46, is taken from the first angle equation.

The first term written in the second equation is the sum of the squares of quantities in the second column of correlates, the next term is the sum of products from the second and third, followed by the term obtained from the second and fourth columns. The η term is taken from the second angle equation. The same scheme is applied to the remaining equations.

The Σ term for the first normal equation is obtained by adding the first horizontal row including η. The Σ term of the second normal equation is obtained by reading down the second column until it terminates then to the right as far as possible, including the η term. The same scheme is applied to the remaining equations. Thus we have -

+4 +2 +2 -3.97 -1.46 = +2.57
+2 +6 -2 +5.69 -1.64 = +10.05
+2 -2 +6 -8.52 +1.78 = -0.74
-3.97 +5.69 -8.52 +41.0098 -6.39 = +27.8198

The Σ terms may be checked independently as follows. For a given normal equation, obtain the sum of the products of quantities in the corresponding column of correlates by those in the same horizontal rows of the Σ column of the correlate equations. This sum should be the same as the sum of the terms of the normal equation preceding the η term. To illustrate for the first equation:

(equation) +4 +2 +2 -3.97 = +4.03, or plus η term = +2.57
(check) -0.15 -0.53 +2.94 +1.77 = +4.03, or plus η term = +2.57

It may be seen that these schemes are algebraically equivalent. This check should always be carried out before attempting to solve the normal equations.

Forward Solution

The forward and back solutions of the normal equations should give us a set of constants from which the v's may be derived. The forward solution is carried out as follows: Write the first normal equation on the first row. On the second row write C_1 followed by the respective quotients, with reversed algebraic sign, of the first term of row 1 divided into each succeeding term. This divided equation expresses C_1 in terms of the other constants and the η term. That is C_1 = -0.50000 C_2 -0.50000 C_3 +0.99250 C_4 +0.36500. The divided equation is checked by obtaining the sum of the terms up to and including η and adding in a -1. This sum should be equal to the Σ term. A horizontal line is then drawn to indicate that this much of the solution is finished.

The solution of the second normal equation may now be carried out. Write the equation, +6 -2 +5.69 -1.64 +10.05 beneath the line in the third row beginning with column 2. To form equation marked (1) which is placed in row 4 beginning with column 2, use the first term of the first divided equation, -0.50000, found in row 2, column 2, to multiply the terms in row 1 beginning with the term, +2, just above the constant factor, and continuing with all subsequent terms to the right. That is, equation (1) is:

-0.50000(+2 +2 -3.97 -1.46 +2.57)
or
-1.0000 -1.0000 +1.9850 +0.7300 -1.2850

Next add the quantities found in the third and fourth rows column by column. We obtain +5.0000 -3.0000 +7.6750 -0.9100 +8.7650, which is written in row 5. The sum of the terms of this equation, added across, up to and including the η term should be equal to the Σ term except for a possible discrepancy in the last figure due to dropping decimals. In row 6 we write C_2 in column 2 followed by the quotients resulting from dividing the first term in row 5, +5.0000, into those that follow, with reversed algebraic signs. This divided equation which completes the solution of the second normal equation is checked by adding the terms across up to and including the η term and adding in a -1. The sum should be equal to the Σ term.

The third normal equation, +6 -8.52 +1.78 -0.74, is written in row 7 beginning with column 3, and just below the line terminating the solution of the second equation. To obtain equation marked (1) in row 8, take -0.50000, found in row 2, column 3, as a constant factor, to multiply the terms in line 1, beginning with column 3. To obtain equation marked (2) in row 9, take +0.60000, found in row 6, column 3, as a constant factor to multiply the terms in row 5 beginning with column 3. Note that in the formation of these multiplied equations, the constant factors are always the terms of the divided equations found in the same column as the number of the normal equation being solved. The terms multiplied are in each case those of the equations found in the respective rows immediately preceding the divided equations from which the constant factors are taken. The multiplication begins in each case with the terms in the column containing the constant factors and continues with all subsequent terms to the right. The equation in row 10 results from the addition column by column of the terms in rows 7, 8 and 9. It is checked in the same manner as the equation in row 5. The divided equation in row 11 is obtained by dividing +3.2000, the first term in row 10, into those that follow with reversed sign. It is checked in the same way as the other divided equations in rows 2 and 6. The horizontal line indicates completion of the solution of the third normal equation.

The fourth and final normal equation, +41.0098 -6.39 +27.8198, is written in row 12 beginning with column 4. The constant factors used to obtain equations (1), (2), and (3), found in rows 13, 14 and 15, are +0.99250, -1.53500, and +0.60312 which are found in the fourth column and in the rows containing the divided equations, viz. 2, 6, and 11. The respective constants are multiplied by the terms in rows 1, 5, and 10, beginning in each case with the term in column 4, that is, the term just above the constant. Rows 12 to 15, inclusive, are added column by column to give the equation in row 16, which is checked in the same fashion as those in rows 5 and 10. The final divided equation beginning with C_4 is obtained by dividing +24.1245, the first term in row 16, into those that follow with reversed sign. As we should expect the first term of the final divided equation, +0.21794 added to -1 gives a sum equal to the last term. The actual value of the constant C_4 is +0.21794. The forward solution is thus completed.

Back Solution

The method of obtaining the back solution should be understood from the following scheme of writing the equations which give the implicit relationship between the constants.

(1) $C_4 = +0.21794$

(2) $C_3 = -0.61375 + 0.60312\ C_4$

(3) $C_2 = +0.18200 - 1.53500\ C_4 + 0.60000\ C_3$

(4) $C_1 = +0.36500 + 0.99250\ C_4 - 0.50000\ C_3 - 0.50000\ C_2$

These equations are obtained by rewriting the divided equations in rows 17, 11, 6 and 2, respectively, of the forward solution, rearranging the terms to read from the η term to the left. It is obvious that these equations must be solved in the order given by eliminating one unknown at a time. The back solution is a formal presentation of the algebraic solution of these equations, consisting of as many columns as there are C's, in this case four, numbered in reverse order. Columns 4, 3, 2, 1 of back solution are exactly the same as equations (1), (2), (3), and (4), written above, except that the indicated multiplications have been carried out. Similarly if we treat the assembly of equations (1), (2), (3), (4) as an array of rows and columns, the column containing the constant terms just to the right of the equality signs, read downward is the same as row 1 of the back solution. The next column containing the products of C_4 is the same as row 2, etc. The details of setting up the back solution and eliminating the constants follow.

The first row contains the terms of the divided equations found in the η column of the forward solution, rows 17, 11, 6 and 2, read upwards and beginning with C_4. The second row beginning in the second column numbered 3 consists of the products of the terms of the divided equations found in the fourth column of the forward solution, viz. +0.60312, -1.53500, +0.99250, taken in the order written by C_4. The quantities in column numbered 3 are added algebraically to give C_3, -0.48231. C_3 is multiplied by the terms of the divided equations found in column 3 of the forward solution, +0.60000, and -0.50000, and the products placed in columns 2 and 1 of the third row of the back solution. Column 2 is added to give C_2, -0.44193. C_2 is multiplied by the term of the first divided equation found in column 2 of the forward solution -0.50000, and the product placed in column numbered 1, row 4, of the back solution. This column is added to give C_1, +1.04343.

v's

The C's are now substituted in the correlate equations to give the v's.

$$\begin{aligned} v_1 &= (-1 \times C_1) + (1\text{-}1 \times C_2) + (2.15 \times C_4) \\ &= -1.04343 + 0.44193 + 0.46857 \\ &= -0.13293 \\ v_2 &= (+1 \times C_2) + (+3.43 \times C_4) \\ &= -0.44193 + 0.74753 \\ &= +0.30560 \end{aligned}$$

This work is presented in tabular form. (see page 11) The v's are adopted to the same number of decimals retained in the observed directions. The adoption is usually simply a matter of rounding off the decimals but the last figure may differ by 1 from the value so obtained, due to the fact that the sums of the angle corrections must equal the closures of the respective triangles. For instance, v_2, +0.30560, is adopted to +0.30 rather than +0.31 as would be expected.

The angle corrections are next determined and entered in the triangle computation. The correction to a given angle is the algebraic sum of the corrections to the directions of the lines which form the angle, taking into account the algebraic signs of the v's, or designations. Let us take as an illustration the computation of the triangle, Omar Ecc., Best, Brunson, as found at the beginning of the triangle form. The correction to the angle at Omar Ecc., designated -8 +10, becomes, after substitution of the v's, -(+0.44) +(-0.48) = -0.92. This correction is entered in column 4. Similarly the correction to the angle at Best, designated +2, is +(+0.30), and the correction to the angle at Brunson, designated -6, is -(+0.98) or -0.98. Note that the sum of these corrections in column 4 is equal to the closure previously determined, -1.60. The closures will be found equal to the sum of the corrections to within the limits of adoption of v's just discussed, if the solution is correct. The spherical angles are then entered and the distributed spherical excess subtracted to give the plane angles which should sum to 180° exactly. The computation of the sides of the triangles is completed. Sides common to different triangles should agree in the logarithms or differ by an amount not exceeding the tabular difference corresponding to 1 in the last decimal place of the smallest angle involved in the calculation. In a first order computation as given in the example the logarithms of the angle functions are carried to eight places and the lengths adopted to seven. In the event common sides differ by an amount within the allowable limit as just discussed, the length derived from the triangle containing the largest length angle is adopted. A satisfactory check in the lengths of common sides is the final criterion of correctness of the whole computation.

The v's or corrections to directions are finally applied in the lists of directions and the adjusted directions entered. As a final check it should be noted whether the use of the adjusted directions will give the spherical angles in the triangle form. This operation is indicated by check marks on the spherical angles.

The foregoing illustrates the adjustment in detail of a quadrilateral. The reader should now consult Special Publication No. 138 for further development of this problem in seeing how an arc of triangulation is adjusted.